Chapter 8:

single variable calculus:

differential calculus

e.g. rates of change
slope of tangent line

integral calculus

e.g. area under the graph of a function

Fundamental theorem of calculus

$$\boxed{\int_a^b f'(x)\,dx = f(b) - f(a)}$$

Remainder of the quarter:

Connecting vector integral calculus (line integrals, surface integrals...) and vector differential calculus (tangent planes, taylor's formula...)

$\Longrightarrow$

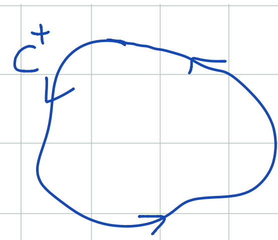
- Green's theorem
- Stoke's theorem
- Gauss's theorem

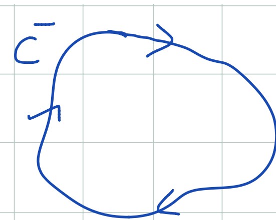
—x—

Green's theorem (8.1)

Relates line integral along a closed curve C in $\mathbb{R}^2$ to a double integral over the region enclose by C.

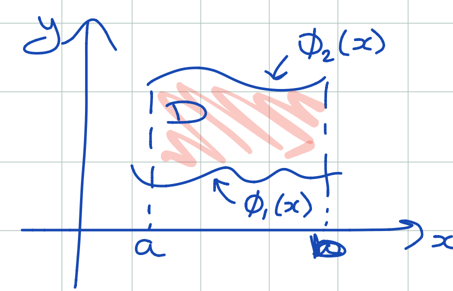
Counter-clockwise



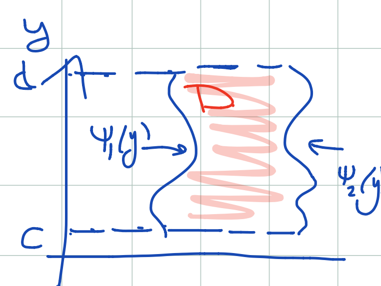


Clockwise

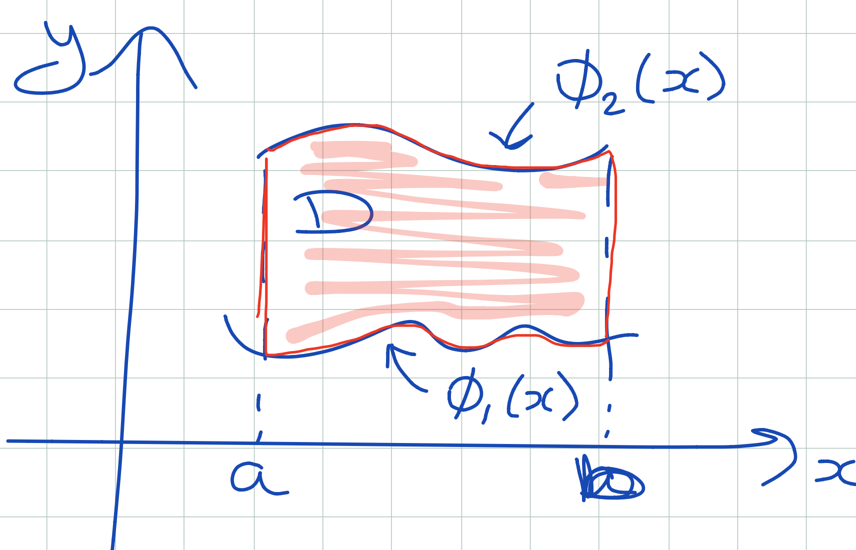
Recall: y-simple region D



x-simple region D



Let's take a y-simple region D, with boundary C and let $P: D \to \mathbb{R}$ be a function



So we can write

$$\iint_D \frac{\partial P}{\partial y}\Big|_{(x,y)} dx\,dy = \underbrace{\int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\Big|_{(x,y)} dy\,dx}_{\text{by the FTC}}$$

(*)

$$= \int_a^b \overbrace{P(x, \phi_2(x)) - P(x, \phi_1(x))}\,dx$$

but $(x, \phi_2(x))$, $a \le x \le b$ is the parametrization of the top part of C going from a to b

and $(x, \phi_1(x))$, $a \le x \le b$ is the parametrization of the bottom part of C going from a to b

So $$\int_a^b P(x, \phi_2(x))\,dx = \int_{C(\text{top},\, a\to b)} P(x,y)\,dx \quad (I)$$

and $$\int_a^b P(x, \phi_1(x))\,dx = \int_{C(\text{bottom},\, a\to b)} P(x,y)\,dx$$

$$= -\int_{C(\text{bottom},\, b\to a)} P(x,y)\,dx \quad (II)$$

Also $$\int_{C(\text{left})} P(x,y)\,dx = \int_{C(\text{right})} P(x,y)\,dx = 0 \quad (III)$$

because x is constant

So:

$$\iint_D \frac{\partial P}{\partial y}\,dx\,dy = \int_{C^-} P\,dx = -\int_{C^+} P\,dx$$

Substituting (I), (II) & (III) in (*)

For a y-simple region D

Similarly:

$$\iint_D \frac{\partial Q}{\partial x}\,dx\,dy = \int_{C^+} Q\,dy = -\int_{C^-} Q\,dy$$

For an x-simple region D

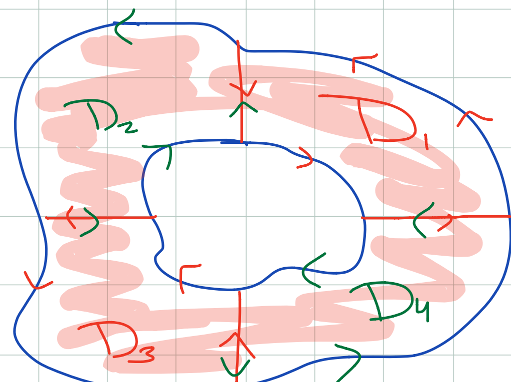
i.e., both x-simple & y-simple

Green's Theorem: Let D be a simple region and let C be its boundary. Let $P: D \subset \mathbb{R}^2 \to \mathbb{R}$ & $Q: D \subset \mathbb{R}^2 \to \mathbb{R}$ have continuous partials.

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy = \int_{C^+} P\,dx + Q\,dy$$

counter clockwise

If the region is not simple, we can break it up into simple regions and sum things up.



Note: As you traverse the boundary, the region should be to your left

Example: Evaluate using Green's theorem

$$\int_{C^+} \underbrace{y^3}_{P}\,dx - \underbrace{x^3}_{Q}\,dy \quad \text{where } C \text{ is the unit circle}$$

$$\int_{C^+} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_{\substack{\text{unit}\\\text{Disk}}} -3x^2 - 3y^2 \,dx\,dy$$

go to polar coord.

$$= \int_0^{2\pi}\int_0^1 -3r^2\, r\,dr\,d\theta = \int_0^{2\pi} -\frac{3}{4}(1^4 - 0)\,d\theta$$

$$= -\frac{6\pi}{4}$$

Verify this by evaluating $\int_{C^+} y^3\,dx - x^3\,dy$ directly

$$\int_{C^+} y^3\,dx - x^3\,dy = \int_0^{2\pi} \left((\sin\theta)^3, -(\cos\theta)^3\right)\cdot\left((\cos\theta)', \sin\theta'\right) d\theta$$

$$= \int_0^{2\pi} -\sin^4\theta - \cos^4\theta \, d\theta = \underset{\text{trig identities}}{\dots} = -6\pi/4$$

Important application of Green's Theorem (Area of a region)

$$\iint_D dx\,dy = \boxed{A = \frac{1}{2}\iint_{\partial D} x\,dy - y\,dx}$$

∂D boundary of D, oriented counterclockwise

Proof: $$\iint_D dx\,dy = \frac{1}{2}\iint_D (1+1)\,dx\,dy = \frac{1}{2}\iint_D \frac{\partial x}{\partial x} - \left(-\frac{\partial y}{\partial y}\right) dx\,dy$$

$$\underset{\text{Green's theorem}}{=} \frac{1}{2}\iint_{\partial D} x\,dy - y\,dx$$

Example

Compute the area of the hypercycloid $S: (\underbrace{a\cos^3\theta, a\sin^3\theta}_{c(\theta)}), 0 \le \theta \le 2\pi$

$(x^{2/3} + y^{2/3} = a^{2/3})$

$$A = \iint_{\partial D} -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} (-a\sin^3\theta, a\cos^3\theta)\cdot\overbrace{(-3a\cos^2\theta\sin\theta, 3a\sin^2\theta\cos\theta)}^{c'(\theta)}\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(3a^2\sin^4\theta\cos^2\theta + 3a^2\sin^2\theta\cos^4\theta\right)d\theta = \frac{3}{2}a^2\int_0^{2\pi}\underbrace{\sin^2\theta\cos^2\theta}_{\left(\frac{\sin 2\theta}{2}\right)^2}\,d\theta$$

bec $(\sin\alpha)^2 = \frac{1-\cos 2\alpha}{2} \longrightarrow \underbrace{\left(\frac{\sin 2\theta}{2}\right)^2}_{\frac{1}{4}\cdot\frac{1-\cos 4\theta}{2}}$

$$= \frac{3}{8}a^2\int_0^{2\pi}\frac{1-\cos 4\theta}{2}\,d\theta = \frac{3}{8}\pi a^2 \quad \blacksquare$$

Curl of a vector field

Del operator $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$

$\Longrightarrow$ gradient: $\nabla f = \vec{i}\frac{\partial f}{\partial x} + \vec{j}\frac{\partial f}{\partial y} + \vec{k}\frac{\partial f}{\partial z}$.

$(f : \mathbb{R}^3 \to \mathbb{R})$

curl: $\nabla\times\vec{F} := \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$(\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3)$

$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}$$

Example: Compute the curl of $\vec{F} = y\vec{i} - x\vec{j}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = +\frac{\partial x}{\partial z}\vec{i} + \frac{\partial y}{\partial z}\vec{j} + \left(\frac{\partial(-x)}{\partial x} - \frac{\partial(y)}{\partial y}\right)\vec{k}$$

$$= 0\vec{i} + 0\vec{j} + (-2)\vec{k}.$$

Curl is associated with rotations. So, informally, if you "drop" a small object with sides dx, dy, dz into a vector field $\vec{F}$, it would rotate <u>about</u> it's axis with angular velocity $\frac{1}{2}(\nabla \times \vec{F})$. See book/more on this later.

A vector field with $\nabla \times \vec{F} = 0$ is called <u>irrotational</u>.

Vector Form of Green's Theorem

Suppose $\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$
and suppose we have a "nice" region D with boundary ∂D (positively oriented).

Then $\displaystyle\int_{\partial D} P\,dx + Q\,dy \quad = \int_{\partial D} \vec{F} \cdot d\vec{s}$

$$\| \qquad\qquad\qquad \|$$

Moreover, $\displaystyle\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dx\,dy = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\,dx\,dy$

bec $\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P(x,y) & Q(x,y) & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$

which implies $(\nabla \times \vec{F}) \cdot \vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

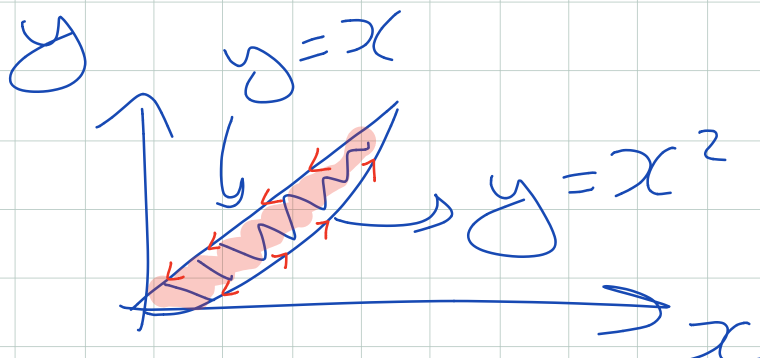
So: $$\iint_{\partial D} \vec{F} \cdot d\vec{s} = \iint_D (\nabla \times \vec{F}) \cdot \vec{k}\,dx\,dy$$

Vector form of Green's Theorem!

Example: Let $\vec{F} = (xy, x-y)$

Compute $\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx\,dy$ where D is the region in the first quadrant bounded by the curves $y = x^2$ & $y = x$

by Green's theorem $\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx\,dy = \int_{\partial D} \vec{F} \cdot d\vec{s}$



First: along the parabola from $x=0$ to $x=1$ (parametrized as $\vec{C}(x): (x, x^2)$, $0 \le x \le 1$)

$$\int_0^1 (F_1, F_2) \cdot (1, 2x)\, dx = \int_0^1 (x, x^2) \cdot (1, 2x)\, dx = \int_0^1 x + 2x^3\, dx$$
$$= \tfrac{1}{2} + \tfrac{2}{4} = 1$$

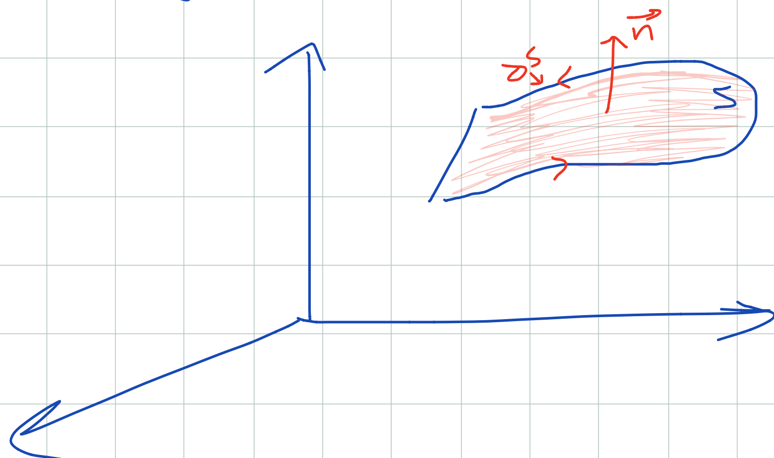
Second: along the line $y = x$ from $x=1$ to $x=0$

$$\int_1^0 (F_1, F_2) \cdot (1, 1)\, dx = -\int_0^1 2x\, dx = -\frac{x^2}{2}\Big|_0^1 = -\frac{1}{2}.$$

$$\Rightarrow \int_{\partial D} \vec{F} \cdot d\vec{s} = 1 - \frac{1}{2} = \frac{1}{2} = \iint_D (\vec{\nabla} \times \vec{F}) \cdot \vec{k}\, dx\,dy.$$

Stoke's theorem (8.2)

Green's theorem dealt with $\overset{(\text{flat})}{\text{Planar}}$ regions.
Stoke's theorem relates the line integral of a vector field around a simple closed curve C to an integral over a surface S with C as its boundary.



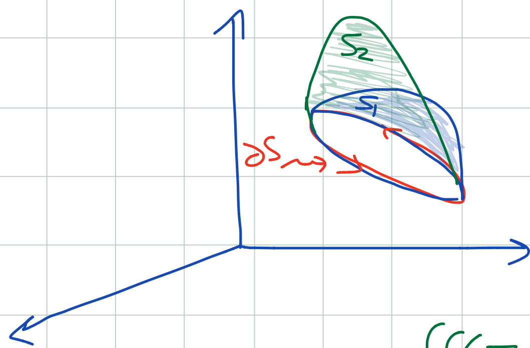
Stokes' theorem

"If C is a closed curve in space, and S is any surface bounded by C, then

$$\int_C \vec{F}\cdot d\vec{s} = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

Rule to determine orientation: Walking along C in the +ve direction, with S to your left, $\vec{n}$ should be pointing up.

Observation: It doesn't matter what S is as long as it's boundary is C.



S_1 & S_2 have the same boundary ∂S

so $\displaystyle\int_{\partial S} \vec{F}\cdot d\vec{s}$

$$= \iint_{S_1} (\nabla \times \vec{F})\cdot d\vec{S}_1 = \iint_{S_2} (\nabla\times\vec{F})\cdot d\vec{S}_2 .$$

Example: Verify Stokes theorem for

$\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$, C = unit circle in the xy-plane oriented ccw $\rightsquigarrow (\cos\theta, \sin\theta, 0)$

$S: z = 1 - x^2 - y^2$

$$\int_{C^+} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} \vec{F}\cdot c'(\theta)\, d\theta = \int_0^{2\pi} (0, \cos\theta, \sin\theta)\cdot(-\sin\theta, \cos\theta, 0)\, d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \frac{1+\cos 2\theta}{2}\, d\theta = \pi .$$

On the other hand

$$\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = ? \qquad \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y \end{vmatrix} = \vec{i} + \vec{j} + \vec{k}$$

Since $S: (x, y, \underbrace{1-x^2-y^2}_{g})$

then $\vec{T}_x \times \vec{T}_y = \left(-\frac{\partial g}{\partial x}, -\frac{\partial g}{\partial y}, 1\right) = (+2x, +2y, 1) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$

$$\text{So } \iint_S (\nabla \times \vec{F}) \cdot \underbrace{d\vec{S}}_{} \overset{(\vec{T}_x \times \vec{T}_y)\,dx\,dy}{} = \iint_{\substack{\text{unit}\\ \text{disk}}} 2x+2y+1 \, dx\,dy$$

$$= \int_0^1 \int_0^{2\pi} (2r\cos\theta + 2r\sin\theta + 1)\, r \, d\theta\, dr$$

$$= \int_0^1 0 + 0 + 2\pi r \, dr = \pi \quad \blacksquare$$

Example: Let S be a surface whose boundary is the circle $x^2+y^2=1$, where S lies above the xy plane with normal vector having +ve $\vec{k}$ component.

Let $\vec{F} = y\vec{i} - x\vec{j} + e^{xz}\vec{k}$ and compute $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$

Soln: By Stoke's theorem $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$

$= \int_C \vec{F} \cdot d\vec{s}$ where $C: (\cos\theta, \sin\theta, 0)$ and $0 \le \theta \le 2\pi$
(hence oriented ccw)

$$= \int_0^{2\pi} \underbrace{(\sin\theta, -\cos\theta, e^0)}_{\vec{F}(\vec{c}(\theta))} \cdot \underbrace{(-\sin\theta, \cos\theta, 0)}_{\vec{c}\,'(\theta)} \, d\theta = \int_0^{2\pi} -1 \, d\theta = -2\pi$$

Note: we didn't care what S was explicitly.

Important Remark: For a surface S with no boundary $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$

e.g. the sphere

by Stoke's theorem!.

— x —

Gauss's theorem (8.4)

But first: Divergence of a vector field.

If $\vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$, then

$$\operatorname{div}(F) = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial}{\partial y} F_2 + \frac{\partial F_3}{\partial z}$$

(dot product of ∇ with $\vec{F}$)

Note: divergence of $\vec{F}$ is a scalar

Example: Compute the divergence of

$$\vec{F} = x^2 \vec{i} + xyz \vec{j} + y \vec{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(y) = 2x + xz + 0$$

Physical interpretation: If $\vec{F}$ is the velocity field of a fluid, then $\nabla\cdot\vec{F}$ is the rate of expansion per unit volume under the flow of the gas

Theorem: $\nabla\cdot(\nabla\times\vec{F})=0$

exercise: check this using the definitions

Gauss's theorem: If S is a closed surface bounding a region W, with normal pointing outward, and if $\vec{F}$ is a vector field defined over W and differentiable:

$$\iint_{\underset{\to \partial W}{S}} \overbrace{\vec{F}\cdot d\vec{S}}^{\vec{F}\cdot\vec{n}\,dS} = \iiint_W \nabla\cdot\vec{F}\,\underbrace{dV}_{dx\,dy\,dz}$$

Example: $\vec{F} = 2x\vec{i} + y^2\vec{j} + z^2\vec{k}$

S: unit sphere. Evaluate

$$\iint_S \vec{F}\cdot d\vec{S}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \nabla\cdot\vec{F}\,dV = \iiint_{\text{"filled in sphere"} = \text{Ball}} \overbrace{(2+2y+2z)}^{\frac{\partial}{\partial x}(2x)+\frac{\partial}{\partial y}(y^2)+\frac{\partial}{\partial z}(z^2)}\,dx\,dy\,dz$$

$$= 2\iiint_{\text{Ball}} dV + 2\cancel{\iiint_{\text{Ball}} y\,dV}_{\to 0} + 2\cancel{\iiint_{B} z\,dV}_{\to 0} = \frac{8}{3}\pi(1)^2$$

Interpretation

At a point (x,y,z), $(\nabla\cdot\vec{F})\big|_{(x,y,z)}$ is the rate of outward flow (flux) per unit volume (or expansion)

So, in a sense, Gauss's theorem tells us that the total outward flow $\iiint_W (\nabla\cdot\vec{F})\, dV$ equals the total flux out of the boundary.

Example Compute $\iint_S \vec{F}\cdot d\vec{S}$ where

S is the surface of the box: $0\le x\le 1$, $0\le y\le 1$, $0\le z\le 1$

and $\vec{F} = (3x + e^{\sqrt{yz}})\vec{i} + (y^2 + 5\sinh(z+x))\vec{j} + (xz + y^{72})\vec{k}$

Sol'n by Gauss's theorem $\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \nabla\cdot\vec{F}\, dV$

$$= \int_0^1\int_0^1\int_0^1 \nabla\cdot\vec{F}\, dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 (3+2y+x)\, dx\,dy\,dz = \int_0^1\int_0^1 \left(3+2y+\tfrac{1}{2}\right) dy\,dz$$
$$= 3.5 + 1 = 4.5$$

$$\frac{\partial}{\partial x}(3x + e^{\sqrt{yz}}) + \frac{\partial}{\partial y}(y^2 + 5\sinh(z+x)) + \frac{\partial}{\partial z}(xz + y^{72})$$

What made this a Gauss's theorem problem?
Field was terrible but $\nabla\cdot\vec{F}$ was nice.

Applications (Physics)

Given a charge density $\rho(x,y,z)$ in a region W, the field $\vec{E}$ satisfies $\nabla\cdot\vec{E} = \rho$ (given)

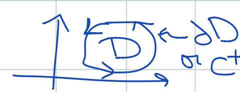
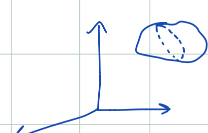
$$\Rightarrow \underbrace{\iiint_W \underbrace{\nabla\cdot\vec{E}}_{\rho}\, dV}_{\text{total charge } Q \text{ inside } W} = \underbrace{\iint_S \vec{E}\cdot d\vec{S}}_{\text{flux out of the surface}}$$

Remark: A two dimensional version of Gauss divergence theorem, i.e., with $\vec{F} = P\vec{i} + Q\vec{j}$ and region D bounded by closed curve $C = \partial D$ is

$$\iint_D (\nabla\cdot\vec{F})\, dx\, dy = \int_C (\vec{F}\cdot\vec{n})\, ds$$

In fact, this is the divergence form of Green's theorem

Review of main theorems

Theorem	Applies in	Applies to	States that
FTLI	2D & 3D	Curves / boundary of curve 	$\int_C \nabla f \cdot d\vec{s} = f(c(b)) - f(c(a))$
Green's Theorem	2D	Regions D / their boundary ∂D 	$\iint_D (\nabla \times \vec{F}) \cdot \vec{k}\, dx\, dy = \int_{\partial D} \vec{F} \cdot d\vec{s}$ $\Leftrightarrow \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\partial D} P\, dx + Q\, dy$ $\Leftrightarrow \iint_D (\nabla \cdot \vec{F})\, dx\, dy = \int_{\partial D} \vec{F} \cdot \vec{n}\, ds$
Stokes's Theorem	3D	Surfaces S / their boundary ∂S 	$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$ $\Leftrightarrow \iint_S (\nabla \times \vec{F}) \cdot \vec{n}\, dS = \int \vec{F} \cdot c'(t)\, dt$
Gauss's theorem	3D	Regions W / their boundary $\partial W = S$ 	$\iiint_W (\nabla \cdot \vec{F})\, dV = \iint_{\partial W} \vec{F} \cdot d\vec{S}$

Conservative Vector Fields (8.3)

FTLI: $\int_{\vec{c}} (\nabla f)\cdot d\vec{s} = f(\vec{c}(b)) - f(\vec{c}(a))$

If the field $\vec{F}$ is a gradient vector field i.e. $\vec{F} = \nabla f$ for some function $f(x,y,z)$ then the line integral is path independent.

example

$\vec{F} = \left(\cos(x)\cos(y) + yze^{xyz},\ -\sin(x)\sin(y) + xze^{xyz},\ xye^{xyz}\right)$

$\vec{c} = \left(\cos(2\pi t)\sin(2\pi t),\ t^2,\ t\right)$, $0 \le t \le 1$

Evaluate $\int_c \vec{F}\cdot d\vec{s}$

Sol'n: $\vec{F} = \nabla f$ where $f = \sin(x)\cos(y) + e^{xyz}$

$\Rightarrow \int_c \vec{F}\cdot d\vec{s} = f(\vec{c}(b)) - f(c(\vec{a}))$

$= f(\cos 2\pi(1)\sin 2\pi(1),\ 1^2,\ 1) - f(\cos 2\pi(0)\sin 2\pi(0),\ 0^2,\ 0)$

$= f(0,1,1) - f(0,0,0) = \sin(0)\cos(1) + e^{0\cdot1\cdot1} - \sin(0)\cos(0) - e^{0\cdot0\cdot0}$

$= 0$

(much easier than $\int_0^1 F\cdot c'(t)\,dt$.)

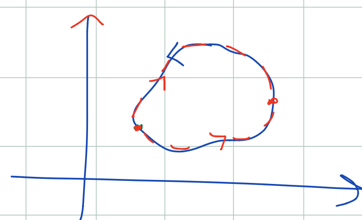
We'd like to know when vector fields are gradients.

Theorem: Let $\vec{F}$ have continuous partial derivatives

All these statements are equivalent:

(i) $\int_C \vec{F}\cdot d\vec{s} = 0$ for all oriented simple closed curves.

(ii) $\int_{C_1} \vec{F}\cdot d\vec{s} = \int_{C_2} \vec{F}\cdot d\vec{s}$ for simple oriented curves with the same endpoints



(iii) $\vec{F} = \nabla f$ for some f

(iv) $\nabla \times \vec{F} = 0$

We call such a field $\vec{F}$ conservative

In the above example, how did we know that $\vec{F}$ was conservative? How did we find f?

Answer: either by inspection 1

or (2) $$\nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \cos(x)\cos(y) + yze^{xyz} & -\sin(x)\sin(y) + xze^{xyz} & xye^{xyz} \end{vmatrix}$$

$$= \left[xe^{xyz} + xy^2ze^{xyz} - \left(xe^{xyz} + x^2yze^{xyz}\right)\right]\vec{i} + \cdots \quad = \vec{0}$$

or (3) If such an f exists: $\frac{\partial f}{\partial x} = \cos x \cos y + yz e^{xyz}$

=

$\frac{\partial f}{\partial y} = -\sin x \sin y + xz e^{xyz}$

$\partial f/\partial z = xy e^{xyz}$

$$\partial f/\partial z = xy e^{xyz} \Rightarrow f(x,y,z) = e^{xyz} + g(x,y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = xz e^{xyz} + \frac{\partial g}{\partial y}$$

So $\partial g/\partial y = -\sin(x)\sin(y) \Rightarrow g(x,y) = \sin(x)\cos(y) + h(x)$

...

So we have $f(x,y,z) = e^{xyz} + \sin(x)\cos y + h(x)$

$$\frac{\partial f}{\partial x} = \cancel{x} yz e^{xyz} + \cos x \cos y + h'(x)$$

So $h'(x) = 0$ works.

Remark: In 2 dimensions (i.e. the planar case)

$$\nabla \times \vec{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

So $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative when

$\partial Q/\partial x = \partial P/\partial y$. (So there is an $f : \nabla f = \vec{F}$)

e.g.

$$\vec{F} = (2xy - \sin x)\vec{i} + x^2\vec{j}$$

is conservative because $\partial Q/\partial x = \frac{\partial(x^2)}{\partial x} = 2x$

$\partial P/\partial y \overset{\shortparallel}{=} 2x$.

To find f we solve $\partial f/\partial x = 2xy - \sin x$, $\frac{\partial f}{\partial y} = x^2$

$$\Rightarrow F(x,y) = x^2 y + \cos x.$$

Example

Let $\vec{c}:[1,2]\longrightarrow\mathbb{R}^2$ be given by $\left(e^{t-1}, \sin(\pi/t)\right)$

Compute $\displaystyle\int_{\vec{c}} 2x\cos(y)\,dx - x^2\sin y\,dy$

This means they want $\displaystyle\int_{\vec{c}}\vec{F}\cdot d\vec{s}$ where $\vec{F}=\left(2x\cos y, -x^2\sin y\right)$

$\partial/\partial y\,(2x\cos y) = -2x\sin y = \partial/\partial x\,(-x^2 y)$

So $\vec{F}$ is conservative & $\displaystyle\int\vec{F}\cdot d\vec{s} = f(\vec{c}(2)) - f(\vec{c}(1))$

$= f(e,1) - f(1,0)$

Moreover $f(x,y) = x^2\cos y$ so $\displaystyle\int_{\vec{c}}\vec{F}\cdot d\vec{s} = e^2\cos(1) - 1$

Recall: $\boxed{\nabla\times(\nabla f) = 0 \quad \& \quad \nabla\times\vec{F}=0 \Rightarrow \vec{F}=\nabla f}$

It is also true that $\boxed{\nabla\cdot(\nabla\times\vec{G}) = 0 \quad \& \quad \nabla\cdot\vec{F}=0 \Rightarrow \vec{F}=\nabla\times\vec{G}}$

has cont's second partials

Maxwell's Equations

- Physically observed laws.
- Relate $\vec{E}(x,y,z,t)$ & $\vec{H}(x,y,z,t)$ the electric & magnetic fields (time varying) to each other and to $\rho(x,y,z,t)$, the charge density and $\vec{J}(x,y,z,t)$ the current density.

The equations

$\nabla \cdot \vec{E} = \rho/\varepsilon_0$ (Gauss's law) (1)

$\nabla \cdot \vec{H} = 0$ (2)

$\nabla \times \vec{E} = -\frac{\partial H}{\partial t}$ (Faraday's law) (3)

$\nabla \times \vec{H} = \mu_0 \left(\varepsilon_0 \partial \vec{E}/\partial t + \vec{J}\right)$ (Ampere's law) (4)

permeability   permittivity

- For example (1) told us that $\iiint_W \rho \, dS$ = total charge $= \iint_{\partial W} \vec{E} \cdot d\vec{S}$
- (2) has no RHS because there are no free magnetic charges
- (3) & (4) tell us that a time varying $\vec{E}$lectric field induces a magnetic field & vice versa.

Let's look at the equations in vacuum i.e. $\vec{J} = 0$, $\rho = 0$

$$\nabla \times \vec{E} = -\frac{\partial H}{\partial t} \quad \& \quad \nabla \times \vec{H} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

So

$$\nabla \times (\nabla \times \vec{E}) = -\frac{\partial (\nabla \times H)}{\partial t} = -\partial/\partial t \left(\mu_0 \varepsilon_0 \partial \vec{E}/\partial t\right) = -\mu_0 \varepsilon_0 \, \partial^2 \vec{E}/\partial t^2$$

$\nabla^2(E_x)\vec{i} + \nabla^2(E_y)\vec{j} + \nabla^2(E_z)\vec{k}$ (vector Laplacian)

but $\nabla \times (\nabla \times \vec{E}) = \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E}$ (see section 4.4)

So $\cancel{\nabla(\nabla \cdot \vec{E})}_{0} - \left(\partial^2 \vec{E}/\partial x^2 + \partial^2 \vec{E}/\partial y^2 + \partial^2 \vec{E}/\partial z^2\right) = -\mu_0 \varepsilon_0 \, \partial^2 \vec{E}/\partial t^2$

$$\Rightarrow \underbrace{(\mu_0 \varepsilon_0)}_{= \frac{1}{c^2}} \partial^2 \vec{E}/\partial t^2 = \nabla^2 \vec{E}$$

Similarly $\frac{1}{c^2} \partial^2 \vec{H}/\partial t^2 = \nabla^2 \vec{H}$

So EM waves propagate at the speed of light and light is an EM wave!! (Maxwell suspected)